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Potential Limits of Frequency Division Multiplexing of N–OFDM Signals Based on Hartley's Basis Functions

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Abstract—The limiting capabilities of the frequency division multiplexing of signals generated by using the N–OFDM method based on the Hartley transform have been studied. The validity of the appropriate Cramer–Rao lower bound for estimates of signal amplitudes was examined by simulation modeling.

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Communication line capacity can be enhanced by using the method of non-orthogonal discrete frequency modulation (N–OFDM) based on frequency division multiplexing of channels by way of transmitting the carriers on non-orthogonal frequencies [1, 2]. Realization of this method by using the classic Fourier transform gives rise to a number of difficulties among which one should note the computational complexity taking into account the complex form of representation of numbers. The application of the Hartley transform (HT) makes it possible to renounce the complex form of data recording and simplify the hardware implementation of the N–OFDM method [3].

The purpose of this study is to determine the potential limits of frequency division multiplexing of signals formed by the N–OFDM method on the basis of the Hartley transform with the probability of the correct demodulation of message equal to 0.9973.

In order to determine the potential limits of frequency division multiplexing of N–OFDM signals, we performed a computational experiment based on using the simulation program developed in the Delphi 7 environment. In essence the experiment consisted of the following steps. A message of fixed length was transformed into a sequence of decimal symbols $A = [a_1 \quad a_2 \quad \dots \quad a_M]^T$ used as amplitudes of signals of different carriers. Next, sampling W of T temporal indications of voltages of the signal mixture to be transmitted at M frequencies [3] was simulated:

$$W = P \cdot A = \begin{bmatrix} \mathrm{cas}\,\omega_1(s_1 - z_1)\Delta t & \mathrm{cas}\,\omega_2(s_1 - z_2)\Delta t & \cdots & \mathrm{cas}\,\omega_M(s_1 - z_M)\Delta t \\ \mathrm{cas}\,\omega_1(s_2 - z_1)\Delta t & \mathrm{cas}\,\omega_2(s_1 - z_2)\Delta t & \cdots & \mathrm{cas}\,\omega_M(s_2 - z_M)\Delta t \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{cas}\,\omega_1(s_T - z_1)\Delta t & \mathrm{cas}\,\omega_2(s_T - z_2)\Delta t & \vdots & \mathrm{cas}\,\omega_M(s_T - z_M)\Delta t \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_M \end{bmatrix}, \quad (1)$$

where $\mathrm{cas}\,\omega_M S_{TM} = \cos\omega_M S_{TM} + \sin\omega_M S_{TM}$ is the Hartley function [3]; $\omega_m = 2\pi f_m$; f_m is the frequency of the mth carrier at the output of a digital-to-analog converter (DAC); s_t is the sequence number of the tth temporal reading of the signal sampling; z_m is the shift of the beginning of the sampling formed with respect to zero phase of the mth carrier ($\varphi_m = 2\pi f_m z_m \Delta t$ is the initial phase of the mth carrier); Δt is the clock step of DAC.

In order to ensure the possibility of transmitting N–OFDM signals by DACs having the different capacity (number of bits), the following restriction was introduced

$$|W_{\max}| \le 2^{R-1} - 1, \quad (2)$$

where R is DAC's number of bits.

The multifrequency signal mixture having passed through the propagation medium at the receiving device output can be presented in a similar matrix notation, which is different from expression (1) due to accounting of the effect of additive noise:

$$U = P \cdot A + N, \tag{3}$$

where $N = [n_1 \quad n_2 \quad \dots \quad n_T]^T$ is the vector of indications of noise voltages.

In performing the computational experiment we used the vector of indications of noise voltages having the normal distribution. The calculations were conducted for 100 realizations of the transmission process of a fixed text message.

Optimal demodulation of the received information was achieved by using the least-squares method for estimating the amplitudes of signals [2]:

$$\hat{A} = \{P^T P\}^{-1} P^T U. \tag{4}$$

In order to enhance the convenience in estimating the admissible value of frequency division multiplexing of N–OFDM signals with respect to signals having orthogonal carriers, it is expedient to express the frequency shift of adjacent channels by using the relative value

$$\xi = \frac{\Delta f}{\Delta F}, \tag{5}$$

where Δf is the frequency spacing of non-orthogonal carriers, ΔF is the frequency spacing of orthogonal carriers.

In accordance with paper [4] the potential accuracy of demodulation of N–OFDM signals using HT is characterized by the Cramer–Rao lower bound (CRLB) for estimate variance of amplitudes of signals. For evaluating CRLB the Fisher information matrix is calculated and in this case the last matrix can be written in the form [2]:

$$I = \frac{1}{\sigma_{\mathrm{n}}^2} P^T P, \tag{6}$$

where σ_{n}^2 is the standard deviation (SD) of noise.

Elements of the diagonal of the inverse Fisher information matrix correspond to values of the variance of unbiased estimates of signal amplitudes [4]. Since for practical calculations it is convenient to utilize the value of root-mean-square (standard) deviations of amplitude estimates, the relationship for determining CRLR can be written in the form:

$$\sigma_{a_m} \geq \sqrt{I_{m,m}^{-1}}, \tag{7}$$

where $m = 1, 2, \dots, M$ specifies the element number of the inverse Fisher matrix.

The confidence limits were specified for establishing compliance of the amplitude SD values of N–OFDM signals obtained by simulation modeling with the Cramer-Rao lower bound. According to paper [5] for 100 realizations of the random process at the confidence probability of 0.999 the lower limit of the confidence interval was found to be at the level of 0.808 of CRLB, while the upper limit—at the level of 1.29. The calculation results of the amplitude SD of N–OFDM signals that do not fall outside the limits of the confidence interval are considered as potentially exact.

Demodulation of N–OFDM signals will be error-free, if deviations of estimates of the amplitudes of received signals from the corresponding amplitudes of the transmitted carriers do not exceed half of the value of the intersymbol interval Δa in the specified scale:

$$|\hat{a}_m - a_m| < \frac{K}{2}\Delta a, \tag{8}$$

where Δa is the distance between values of two nearest information symbols expressed in DAC quanta; $\hat{a}_m$, a_m are amplitudes of the received and transmitted N–OFDM signal at the mth carrier; K is the scale factor taking into account nonidentity of intersymbol levels in the transmitting and receiving segments.

Taking into account that the received signal is subjected to noise exposure and amplitudes $\hat{a}_m$ are random, expression (8), in case of a sufficiently large number of tests, can be written in the form

$$|\hat{a}_{m_i} - M(\hat{A}_m)| < \frac{K}{2}\Delta a, \tag{9}$$

where $\hat{a}_{m_i}$ is the amplitude of the received signal at the mth carrier in the ith realization, $M(\hat{A}_m)$ is mathematical expectation of the amplitude of the signal received at the mth carrier.

As follows from expressions (8) and (9), for ensuring the correct reception of N–OFDM signals it is sufficient that deviations of amplitudes of N–OFDM signal carriers from their true values (mathematical expectation) would not fall outside the limits of interval $\pm 0.5K\Delta a$.

The probability of the fact that deviation of the normally distributed random value X from its mathematical expectation $M(X)$ is less than preassigned positive number δ can be determined by the formula [6]:

$$P(|X - M(X)| < \delta) = 2\Phi\left(\frac{\delta}{\sigma}\right), \tag{10}$$

where σ is standard deviation (SD) of random value X, $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-t^2/2}dt$ is the Laplace function or probability integral.

As regards the case under consideration, expression (10) can be rewritten in the form:

$$P(|\hat{a}_{m_1} - M(\hat{A})| < \delta) = 2\Phi\left(\frac{\delta}{\sigma}\right). \tag{11}$$

Using relationship (11) let us calculate the probability of deviation of the normally distributed random value $\hat{a}_{m_i}$ from its mathematical expectation for values of δ equal to σ_{a_m}, $2\sigma_{a_m}$, $3\sigma_{a_m}$, and $4\sigma_{a_m}$ [6]:

$$P(a_{m_i} - M(\hat{A})| < \sigma_{a_m}) = 2\Phi(1) \approx 0.6827,$$

$$P(|a_{m_i} - M(\hat{A})| < 2\sigma_{a_m}) = 2\Phi(2) \approx 0.9545,$$

$$P(|a_{m_i} - M(\hat{A})| < 3\sigma_{a_m}) = 2\Phi(3) \approx 0.9973,$$

$$P(|a_{m_i} - M(\hat{A})| < 4\sigma_{a_m}) = 2\Phi(4) \approx 0.99994.$$

As follows from the results obtained, for ensuring the correct reception of N–OFDM signals with probability 0.9973 it is sufficient that deviations of amplitudes of carriers from their mathematical expectation would not exceed $\pm 3\sigma_{a_m}$.

Thus, the problem of determining the potential boundary of frequency division multiplexing of the M-frequency N–OFDM signal using Hartley's basis functions with probability of the correct reception 0.9973 is reduced to determining the minimum value of ξ for which the following condition is still valid:

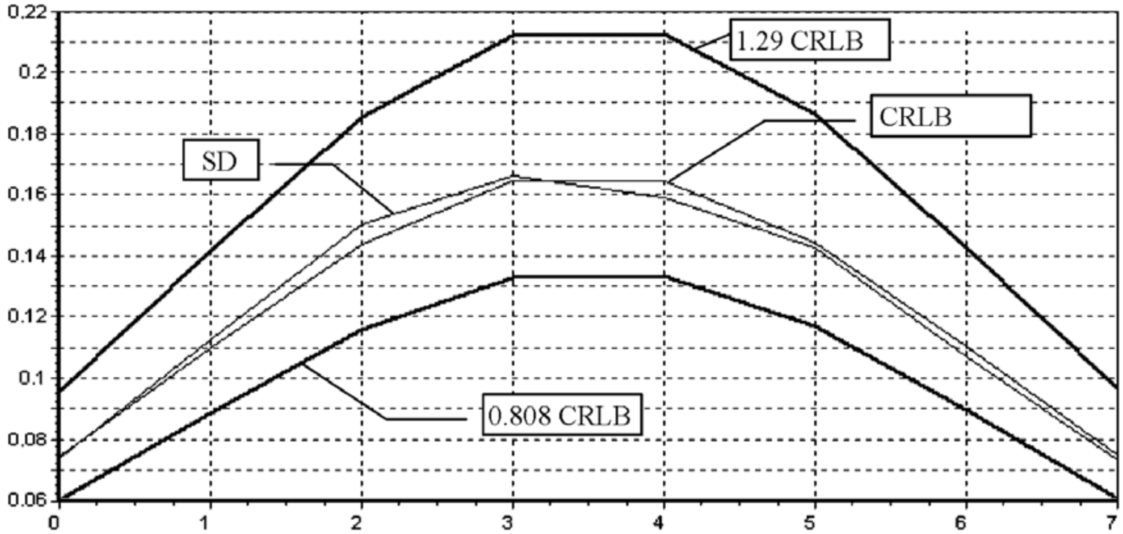

Fig. 1.

$$\Delta a \geq 6\sigma_{a_m}, \tag{12}$$

where σ_{a_m} is the standard deviation of estimates of amplitudes of M signals.

The computing experiment was conducted for 4- and 8-frequency N–OFDM signals with sampling frequency 100 MHz (0.01 μsec) and the different number of temporal samples. The intersymbol interval Δa was selected to ensure the maximum use of DAC dynamic range and the concurrent validity of condition (2).

The plots of measured SD, CRLB, and confidence limits for an 8-frequency N–OFDM signal transmitted with the help of 512 discrete readings are presented in Fig. 1. As can be seen from Fig. 1, the experimental SD values obtained do not fall outside the confidence interval of CRLB, while the curve of SD values approaches the CRLB line. Hence, the results obtained are potentially exact.

Table 1 presents the selected numerical values obtained as a result of simulating SD of estimates of amplitudes of 4- and 8-frequency N–OFDM signals based on Hartley's basis functions, and also the potential boundaries of frequency division multiplexing of subcarriers with the correct reception probability of 0.9973 in terms of 100 realizations. At a smaller value of ξ a higher frequency division multiplexing (smaller frequency spacing) of non-orthogonal signals is achieved. In order to enhance the frequency division multiplexing, it is necessary to increase the number of readings subjected to processing (sampling length) that can be easily explained by the effect of coherent integration of readings of the cas function.

As frequency spacing between subcarriers is reduced, the minimum value of admissible intersymbol interval in terms of amplitude is increased. In calculations using the simulation model it was assumed that the maximum value of the resultant amplitude of the multifrequency signal to be transmitted did not exceed 2047 quanta (for a 12-bit DAC). In this case, the possibility of in-phase summation of cas-functions of all subcarriers was accounted for. Thus, if the maximum amplitude is equal to 255 and 63, the number of data bits that can be transmitted within one signal frame, i.e., the interval of information stability, will be 8 and 6, respectively, etc.

As the number of samples of DAC was increased, it was accompanied by the reduction of CRLB. The reduction of the frequency division multiplexing indicator ξ resulted in the rise of CRLB values.

Generally, the results of simulation confirmed the possibility of reducing the frequency band occupied by an 8-frequency signal packet by more than one third as compared with conventional OFDM-signal (Table 1, Δa is the intersymbol interval).

Thus, the potential limits of frequency division multiplexing of 4- and 8-frequency N–OFDM signals on the basis of Hartley's basis functions were determined. The results obtained can be used in developing the proposals and updating stations of radio relay, troposcatter, and space communications.

Table 1

ξ, %	Δf, kHz	Number of samples	Carrier number*	SD	CRLB	CRLB confidence limits		Maximum admissible amplitude**	Minimum admissible Δa
						lower	upper		
4-frequency N–OFDM signal									
28.6	446.875	64	3	0.33036	0.34969	0.28255	0.4511	255	2
26.6	207.8125	128	3	0.32823	0.30795	0.24882	0.39725		
23.6	92.1875	256	2	0.33256	0.31996	0.25853	0.41274		
18.5	289.0625	64	3	1.32405	1.4608	1.18033	1.88443	63	8
17.3	135.15625	128	3	1.32078	1.25789	1.01637	1.62267		
15.2	59.375	256	2	1.32241	1.3272	1.07238	1.71208		
11.9	185.9375	64	3	5.25321	5.85147	4.72799	7.5484	15	32
11.1	86.71875	128	3	5.22604	5.04331	4.07499	6.50587		
9.7	37.890625	256	2	5.251	5.34576	4.31937	6.89603		
8-frequency N–OFDM signal									
40.2	314.0625	128	5	0.164	0.16908	0.13662	0.21811	255	1
38.7	151.17188	256	4	0.16557	0.15691	0.12678	0.20241		
36.9	72.070313	512	4	0.16613	0.16442	0.13286	0.21211		
33.9	264.84375	128	5	0.6564	0.713	0.57611	0.91977	63	4
32.8	128.125	256	5	0.6571	0.65637	0.53035	0.84672		
31.4	61.328125	512	4	0.65904	0.66633	0.5384	0.85957		
28.8	225	128	5	2.6259	2.84633	2.29984	3.67177	15	16
27.8	108.59375	256	5	2.65987	2.63464	2.12879	3.39868		
26.5	51.757813	512	4	2.6506	2.71764	2.19585	3.50576		

* Calculation results presented that correspond to the carrier with the highest SD.
** Maximum admissible amplitude of separate carriers of N–OFDM signal.

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