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Influence of Jitter in ADC on Precision of Direction-Finding by Digital Antenna Arrays

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Abstract—Direction-finding methods by linear and row-column digital antenna arrays are synthesized. Approximate expressions for dispersions and expected value of angle of arrival estimate are obtained under assumption that jitter is small. Simulation results are presented.

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A desire to decrease the number of frequency conversions in a radio receiving circuitry of radars to increase the system's dynamic range and to decrease intrinsic noise of the receiver leads to higher signal frequency at the receiving circuit's output [1]. This leads to a situation when in digital antenna arrays (DAR), which are essentially multichannel analogue-to-digital conversion systems, the level of digital signal's noise caused by ADC's jitter during sampling increases. Due to this revision of expressions used to calculate precision of direction-finding is required.

Earlier in [2] using simulation of direction-finding by linear DAR by solving the likelihood equation (by enumeration) it was shown estimate's dispersion for the found direction depends on the sum of additive noise and ADC's jitter noise powers under assumption that jitter is small. A modified expression of the lower Kramer–Rao boundary (LKRB) was obtained in [3] for the problem of direction-finding by linear DAR under the presence of ADC's jitter, while in [4] a similar modified expression for LKRB of a planar equidistant row-column DAR is given.

It is more convenient to have analytical expression for direction to the signal source. Hence, when analyzing influence of ADC jitter on precision of direction-finding, in the present work such an expression is synthesized for the cases of linear and row-column arrays. Further using linearization of random variables function [5] approximate expressions are obtained that characterize dispersion and expected value of the synthesized estimates in the presence of ADC jitter.

ESTIMATES SYNTHESIS

To synthesize estimates we'll use ideas described in [6].

Let's consider a linear DAR consisting of N elements (Fig. 1) with element spacing d. Angle β indicating direction to the signal source will be calculated from the normal to the array. Let's assume that the signal with frequency F received by each element is converted to intermediate frequency (IF) f with simultaneous generation of analogue complex signal.

An expression for voltage at the output of nth receiving channel may be represented as

$$\dot{u}_n(t) = A\exp(j(\omega t - \Omega n d c^{-1}\sin\beta + \varphi)), \tag{1}$$

where n is the receiving channel number, A is signal's amplitude, $\omega = 2\pi f$ is circular intermediate frequency, $\Omega = 2\pi F$ is the frequency of received signal, c is the light velocity, φ is the initial phase of signal at the receiving channel's output.

Let's assume that signal at the output of receiving channels is sampled using synchronously clocked ADCs. Expressions for kth sample of co-phase and quadrature components at ADC outputs may be written down as

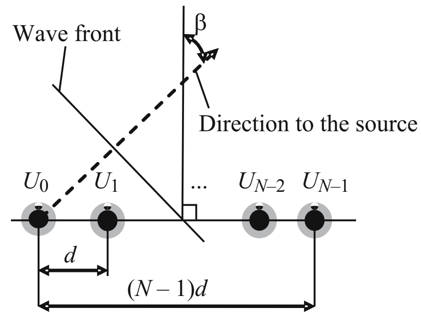

Fig. 1.

$$U^C_{n,k} = A\cos\left(\Theta^C_k - \Psi n \sin\beta\right) + \eta^C_{n,k}, \tag{2}$$

$$U^S_{n,k} = A\sin\left(\Theta^S_k - \Psi n \sin\beta\right) + \eta^S_{n,k}, \tag{3}$$

where $\Theta^C_k = \omega(Tk + \tau^C_{n,k}) + \varphi$, $\Theta^S_k = \omega(Tk + \tau^S_{n,k}) + \varphi$, T is sampling period, $\Psi = \Omega d c^{-1}$, $\tau^{C(S)}_{n,k}$ is random time shift (jitter) during generation of kth sample for nth receiving channel, $\eta^{C(S)}_{n,k}$ is additive noise voltage when generating kth sample for nth receiving channel, C and S are co-phase and quadrature components respectively.

Further we assume that samples $\tau^{C(S)}_{n,k}$ and $\eta^{C(S)}_{n,k}$ along the array's aperture are independent, have zero mean values and dispersion σ^2_τ (for random time shifts) and σ^2_η (for dispersion of additive noise), equal for all ADC in the array.

In absence of noise let's consider the following sums of voltage samples at ADC outputs

$$U^C_{n,k} + U^C_{n+1,k} = \left(1 + \cos\left(\Psi \sin\beta\right)\right) U^C_{n,k} + \sin\left(\Psi \sin\beta\right) U^S_{n,k}, \tag{4}$$

$$U^S_{n,k} + U^S_{n+1,k} = \left(1 + \cos\left(\Psi \sin\beta\right)\right) U^S_{n,k} - \sin\left(\Psi \sin\beta\right) U^C_{n,k}. \tag{5}$$

Expressions (4) and (5) may be also represented as

$$U^C_{n+1,k} - \mu U^C_{n,k} - \nu U^S_{n,k} = 0, \tag{6}$$

$$U^S_{n+1,k} - \mu U^S_{n,k} + \nu U^C_{n,k} = 0, \tag{7}$$

where $\mu = \cos(\Psi \sin\beta)$ and $\nu = \sin(\Psi \sin\beta)$.

To estimate the value of angle β we'll use the least-squares method. Using expressions (6) and (7) we'll write down the target function as

$$F = \sum_{k=0}^{K-1} \sum_{n=0}^{N-2} \left(U^C_{n+1,k} - \mu U^C_{n,k} - \nu U^S_{n,k} \right)^2$$

$$+ \sum_{k=0}^{K-1} \sum_{n=0}^{N-2} \left(U^S_{n+1,k} - \mu U^S_{n,k} + \nu U^C_{n,k} \right)^2. \tag{8}$$

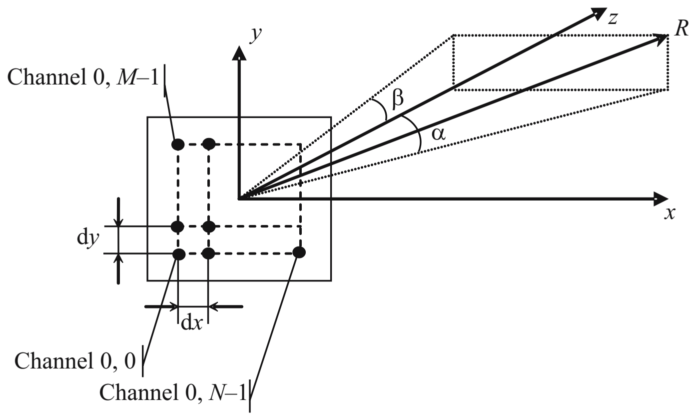

Fig. 2.

Let’s minimize the value of function F. Equating partial derivatives of F with respect to μ and ν to zero and solving the obtained system of equations yields estimates for μ and ν:

$$\mu = \frac{\sum_{k=0}^{K-1}\sum_{n=0}^{N-2} U_{n,k}^{S} U_{n+1,k}^{S} + \sum_{k=0}^{K-1}\sum_{n=0}^{N-2} U_{n,k}^{C} U_{n+1,k}^{C}}{\sum_{k=0}^{K-1}\sum_{n=0}^{N-2}\left(U_{n,k}^{C}\right)^2 + \sum_{k=0}^{K-1}\sum_{n=0}^{N-2}\left(U_{n,k}^{S}\right)^2}, \tag{9}$$

$$\nu = \frac{\sum_{k=0}^{K-1}\sum_{n=0}^{N-2} U_{n,k}^{S} U_{n+1,k}^{C} - \sum_{k=0}^{K-1}\sum_{n=0}^{N-2} U_{n,k}^{C} U_{n+1,k}^{S}}{\sum_{k=0}^{K-1}\sum_{n=0}^{N-2}\left(U_{n,k}^{C}\right)^2 + \sum_{k=0}^{K-1}\sum_{n=0}^{N-2}\left(U_{n,k}^{S}\right)^2}. \tag{10}$$

The estimate of angle β is represented as

$$\hat{\beta} = \arcsin\left(\Psi^{-1}\arctan\left(\frac{\nu}{\mu}\right)\right). \tag{11}$$

Using results of (9) and (10), let’s rewrite expression (11) in the form

$$\hat{\beta} = \arcsin\left(\Psi^{-1}\arctan\left(\frac{\sum_{k=0}^{K-1}\sum_{n=0}^{N-2} U_{n,k}^{S} U_{n+1,k}^{C} - \sum_{k=0}^{K-1}\sum_{n=0}^{N-2} U_{n,k}^{C} U_{n+1,k}^{S}}{\sum_{k=0}^{K-1}\sum_{n=0}^{N-2} U_{n,k}^{S} U_{n+1,k}^{S} + \sum_{k=0}^{K-1}\sum_{n=0}^{N-2} U_{n,k}^{C} U_{n+1,k}^{C}}\right)\right). \tag{12}$$

Let’s generalize the suggested method to the case of rectangular row-column equidistant array with dimensions N by M elements (Fig. 2).

Voltage at instant t at the output of receiving channel may be written down for each antenna element as

$$\dot{u}_{n,m}(t) = A\exp\left(j\left(\omega t - \Omega n\frac{d_x \sin\alpha}{c} - \Omega m\frac{d_y \sin\beta}{c} + \varphi\right)\right), \tag{13}$$

where $n(m)$ is horizontal (vertical) element number, A is signal's amplitude, $\omega = 2\pi f$ is circular intermediate frequency, $\Omega = 2\pi F$ is circular frequency of the received signal, c is the light velocity, φ is initial phase at the receiving channel's output, $d_x(d_y)$ is distance between horizontal (vertical) elements, angles α and β are defined as angles between normal to the antenna array's geometrical center and projection of direction to the signal on the plane xOz (for angle α) and on the plane yOz (for angle β).

Like in the previous case, at the outputs of receiving channels signal is samples using synchronously locked ADCs with jitter. Then expressions for kth sample of co-phase and quadrature components at ADC outputs have the following appearance:

$$U^C_{n,m,k} = A\cos\Big(\Theta^C_{n,m,k} - \Psi n\sin\alpha - \Upsilon m\sin\beta\Big) + \eta^C_{n,m,k}, \tag{14}$$

$$U^S_{n,m,k} = A\sin\Big(\Theta^S_{n,m,k} - \Psi n\sin\alpha - \Upsilon m\sin\beta\Big) + \eta^S_{n,m,k}, \tag{15}$$

where $U^C_{n,m,k}$ and $U^S_{n,m,k}$ are samples of co-phase and quadrature signals at ADC outputs of the receiving channel (n, m), $\Theta^C_{n,m,k} = \omega(Tk + \tau^C_{n,m,k}) + \varphi$, $\Theta^S_{n,m,k} = \omega(Tk + \tau^S_{n,m,k}) + \varphi$, $\Psi = \Omega d_x c^{-1}$, $\Upsilon = \Omega d_y c^{-1}$, T is ADC sampling period, $\tau^C_{n,m,k}$, $\tau^S_{n,m,k}$ and $\eta^C_{n,m,k}$, $\eta^S_{n,m,k}$ are random time shift and additive noise voltage during generation of kth sample of co-phase and quadrature components of the receiving channel (n, m), respectively.

Similarly to the case of linear DAR, neglecting noise, let's consider the following sums of voltage samples:

$$U^C_{n,m,k} + U^C_{n+1,m,k} = \big(1+\cos(\Psi\sin\alpha)\big)U^C_{n,m,k} + \sin(\Psi\sin\alpha)U^S_{n,m,k}, \tag{16}$$

$$U^S_{n,m,k} + U^S_{n+1,m,k} = \big(1+\cos(\Psi\sin\alpha)\big)U^S_{n,m,k} - \sin(\Psi\sin\alpha)U^C_{n,m,k}, \tag{17}$$

$$U^C_{n,m,k} + U^C_{n,m+1,k} = \big(1+\cos(\Upsilon\sin\beta)\big)U^C_{n,m,k} + \sin(\Upsilon\sin\beta)U^S_{n,m,k}, \tag{18}$$

$$U^S_{n,m,k} + U^S_{n,m+1,k} = \big(1+\cos(\Upsilon\sin\beta)\big)U^S_{n,m,k} - \sin(\Upsilon\sin\beta)U^C_{n,m,k}. \tag{19}$$

Let's rewrite expression (16)–(19) in the following form:

$$U^C_{n+1,m,k} - \mu_\alpha U^C_{n,m,k} - \nu_\alpha U^S_{n,m,k} = 0, \tag{20}$$

$$U^S_{n+1,m,k} - \mu_\alpha U^S_{n,m,k} + \nu_\alpha U^C_{n,m,k} = 0, \tag{21}$$

$$U^C_{n,m+1,k} - \mu_\beta U^C_{n,m,k} - \nu_\beta U^S_{n,m,k} = 0, \tag{22}$$

$$U^S_{n,m+1,k} - \mu_\beta U^S_{n,m,k} + \nu_\beta U^C_{n,m,k} = 0, \tag{23}$$

where $\mu_\alpha = \cos(\Psi\sin\alpha)$, $\nu_\alpha = \sin(\Psi\sin\alpha)$, $\mu_\beta = \cos(\Upsilon\sin\beta)$, $\nu_\beta = \sin(\Upsilon\sin\beta)$.

Using expressions (20)–(23) let's write down the target function for determining the values of μ_α, ν_α, μ_β, ν_β using the least-squares method:

$$F_2 = \sum_{k=0}^{K-1}\sum_{n=0}^{N-2}\sum_{m=0}^{M-1}\left(U^C_{n+1,m,k} - \mu_\alpha U^C_{n,m,k} - \nu_\alpha U^S_{n,m,k}\right)^2$$

$$+\sum_{k=0}^{K-1}\sum_{n=0}^{N-2}\sum_{m=0}^{M-1}\left(U^S_{n+1,m,k} - \mu_\alpha U^S_{n,m,k} + \nu_\alpha U^C_{n,m,k}\right)^2$$

$$+\sum_{k=0}^{K-1}\sum_{n=0}^{N-1}\sum_{m=0}^{M-2}\left(U^C_{n,m+1,k} - \mu_\beta U^C_{n,m,k} - \nu_\beta U^S_{n,m,k}\right)^2$$

$$+\sum_{k=0}^{K-1}\sum_{n=0}^{N-1}\sum_{m=0}^{M-2}\left(U^S_{n,m+1,k} - \mu_\beta U^S_{n,m,k} + \nu_\beta U^C_{n,m,k}\right)^2. \tag{24}$$

Let's minimize the value of function F_2. Equating partial derivatives of F_2 with respect to μ_α, ν_α, μ_β, ν_β to zero and solving the obtained system of equations with respect to μ_α, ν_α, μ_β, ν_β yields

$$\mu_\alpha = \frac{\sum_{k=0}^{K-1}\sum_{n=0}^{N-2}\sum_{m=0}^{M-1}\left(U^C_{n,m,k}U^C_{n+1,m,k} + U^S_{n,m,k}U^S_{n+1,m,k}\right)}{\sum_{k=0}^{K-1}\sum_{n=0}^{N-2}\sum_{m=0}^{M-1}\left(\left(U^C_{n,m,k}\right)^2 + \left(U^S_{n,m,k}\right)^2\right)}, \tag{25}$$

$$\nu_\alpha = \frac{\sum_{k=0}^{K-1}\sum_{n=0}^{N-2}\sum_{m=0}^{M-1}\left(U^S_{n,m,k}U^C_{n+1,m,k} - U^C_{n,m,k}U^S_{n+1,m,k}\right)}{\sum_{k=0}^{K-1}\sum_{n=0}^{N-2}\sum_{m=0}^{M-1}\left(\left(U^C_{n,m,k}\right)^2 + \left(U^S_{n,m,k}\right)^2\right)}, \tag{26}$$

$$\mu_\beta = \frac{\sum_{k=0}^{K-1}\sum_{n=0}^{N-1}\sum_{m=0}^{M-2}\left(U^C_{n,m,k}U^C_{n,m+1,k} + U^S_{n,m,k}U^S_{n,m+1,k}\right)}{\sum_{k=0}^{K-1}\sum_{n=0}^{N-1}\sum_{m=0}^{M-2}\left(\left(U^C_{n,m,k}\right)^2 + \left(U^S_{n,m,k}\right)^2\right)}, \tag{27}$$

$$\nu_\beta = \frac{\sum_{k=0}^{K-1}\sum_{n=0}^{N-1}\sum_{m=0}^{M-2}\left(U^S_{n,m,k}U^C_{n,m+1,k} - U^C_{n,m,k}U^S_{n,m+1,k}\right)}{\sum_{k=0}^{K-1}\sum_{n=0}^{N-1}\sum_{m=0}^{M-2}\left(\left(U^C_{n,m,k}\right)^2 + \left(U^S_{n,m,k}\right)^2\right)}. \tag{28}$$

Let's write down the estimates for α and β as follows:

$$\hat{\alpha} = \arcsin\left(\Psi^{-1}\arctan\left(\frac{\nu_\alpha}{\mu_\alpha}\right)\right),$$

$$\hat{\beta} = \arcsin\left(\Upsilon^{-1}\arctan\left(\frac{\nu_\beta}{\mu_\beta}\right)\right). \tag{29}$$

Using results (25)–(28) finally we rewrite estimates (29) to obtain expressions

$$\hat{\alpha} = \arcsin\left(\Psi^{-1}\arctan\left(\frac{\sum_{k=0}^{K-1}\sum_{n=0}^{N-2}\sum_{m=0}^{M-1}\left(U_{n,m,k}^{S}U_{n+1,m,k}^{C} - U_{n,m,k}^{C}U_{n+1,m,k}^{S}\right)}{\sum_{k=0}^{K-1}\sum_{n=0}^{N-2}\sum_{m=0}^{M-1}\left(U_{n,m,k}^{C}U_{n+1,m,k}^{C} + U_{n,m,k}^{S}U_{n+1,m,k}^{S}\right)}\right)\right), \tag{30}$$

$$\hat{\beta} = \arcsin\left(\Upsilon^{-1}\arctan\left(\frac{\sum_{k=0}^{K-1}\sum_{n=0}^{N-1}\sum_{m=0}^{M-2}\left(U_{n,m,k}^{S}U_{n,m+1,k}^{C} - U_{n,m,k}^{C}U_{n,m+1,k}^{S}\right)}{\sum_{k=0}^{K-1}\sum_{n=0}^{N-1}\sum_{m=0}^{M-2}\left(U_{n,m,k}^{C}U_{n,m+1,k}^{C} + U_{n,m,k}^{S}U_{n,m+1,k}^{S}\right)}\right)\right). \tag{31}$$

STATISTICAL ANALYSIS OF ESTIMATES

Estimates (12), (30) and (31) due to influence of noise represent random variables. Let's obtain approximate expressions for expected value and dispersion of the synthesized estimates in conditions of ADC jitter. To accomplish this we'll use the method of linearizing random variable function [5].

Representing the estimate (12) as a function of small random samples of jitter and additive noise yields

$$\hat{\beta} = \hat{\beta}\left(\vec{\eta}, \vec{\tau}\right), \tag{32}$$

where $\vec{\eta} = (\eta_{0,0}^{C}, \eta_{1,0}^{C}, \ldots, \eta_{N-1,K-1}^{C}, \eta_{0,0}^{S}, \eta_{1,0}^{S}, \ldots, \eta_{N-1,K-1}^{S})$ is a vector of additive noise samples, $\vec{\tau} = (\tau_{0,0}^{C}, \tau_{1,0}^{C}, \ldots, \tau_{N-1,K-1}^{C}, \tau_{0,0}^{S}, \tau_{1,0}^{S}, \ldots, \tau_{N-1,K-1}^{S})$ is a vector of random time shifts.

Let's write down approximate values of the estimate as expansion into the Taylor series with respect to parameters $\tau_{n,k}^{C}$, $\tau_{n,k}^{S}$, $\eta_{n,k}^{C}$, $\eta_{n,k}^{S}$ retaining the expansion terms up to the first infinitesimal order inclusively:

$$\hat{\beta} \approx \hat{\beta}\left(\vec{\tau}_0, \vec{\eta}_0\right) + \sum_{k=0}^{K-1}\sum_{n=0}^{N-1}\left.\frac{\partial\hat{\beta}}{\partial\tau_{n,k}^{C}}\right|_{\substack{\vec{\eta}=\vec{\eta}_0,\\ \vec{\tau}=\vec{\tau}_0}}\tau_{n,k}^{C} + \sum_{k=0}^{K-1}\sum_{n=0}^{N-1}\left.\frac{\partial\hat{\beta}}{\partial\tau_{n,k}^{S}}\right|_{\substack{\vec{\eta}=\vec{\eta}_0,\\ \vec{\tau}=\vec{\tau}_0}}\tau_{n,k}^{S}, \tag{33}$$

where $\vec{\eta}_0 = \vec{\eta}\,|_{\eta_{n,k}^{C}=0, \eta_{n,k}^{S}=0}$, $\vec{\tau}_0 = \vec{\tau}\,|_{\eta_{n,k}^{C}=0, \eta_{n,k}^{S}=0}$ $(n = \overline{0, N-1},\ k = \overline{0, K-1})$.

According to the methodology described in [5] the expected value of the estimate has the following appearance:

$$E\{\hat{\beta}\} \approx \hat{\beta}\left(\vec{\tau}_0, \vec{\eta}_0\right) = \beta, \tag{34}$$

where $E\{\ldots\}$ is the expected value calculation operand.

Using expression (33) for dispersion we may write down

$$D\{\hat{\beta}\} = \sigma_{\tau}^{2}\sum_{k=0}^{K-1}\sum_{n=0}^{N-1}\left(\left.\frac{\partial\hat{\beta}}{\partial\tau_{n,k}^{C}}\right|_{\substack{\vec{\eta}=\vec{\eta}_0,\\ \vec{\tau}=\vec{\tau}_0}}\right)^2 + \sigma_{\tau}^{2}\sum_{k=0}^{K-1}\sum_{n=0}^{N-1}\left(\left.\frac{\partial\hat{\beta}}{\partial\tau_{n,k}^{S}}\right|_{\substack{\vec{\eta}=\vec{\eta}_0,\\ \vec{\tau}=\vec{\tau}_0}}\right)^2$$

$$+\sigma_\eta^2 \sum_{k=0}^{K-1}\sum_{n=0}^{N-1}\left(\left.\frac{\partial\hat{\beta}}{\partial\eta_{n,k}^C}\right|_{\substack{\vec{\eta}=\vec{\eta}_0,\\ \vec{\tau}=\vec{\tau}_0}}\right)^2 + \sigma_\eta^2 \sum_{k=0}^{K-1}\sum_{n=0}^{N-1}\left(\left.\frac{\partial\hat{\beta}}{\partial\eta_{n,k}^S}\right|_{\substack{\vec{\eta}=\vec{\eta}_0,\\ \vec{\tau}=\vec{\tau}_0}}\right)^2, \tag{35}$$

where $D\{\ldots\}$ is the dispersion calculation operand.

Calculating the derivatives and substituting them into expression (35) we finally obtain an expression for estimate's dispersion:

$$D\{\hat{\beta}\} = \frac{3}{2}\frac{\omega^2\sigma_\tau^2}{K(N-1)^2\Psi^2\cos^2\beta} + \frac{2\sigma_\eta^2}{A^2K(N-1)^2\Psi^2\cos^2\beta}$$

$$+\frac{1}{2}\frac{\omega^2\sigma_\tau^2}{K^2(N-1)^2\Psi^2\cos^2\beta}\frac{\sin(2\omega TK)}{\sin(2\omega T)}$$

$$\times\cos\big(2\omega T(K-1)+4\varphi-2(N-1)\Psi\sin\beta\big)\cos\big(2(N-1)\Psi\sin\beta\big). \tag{36}$$

In the case when the accumulation period contains an integral number of input signal's periods, i.e. when $fT = n/m$ is irreducible fraction (n and m are integers) and $K = Pm$ (P is integer, $(m > 2) \wedge (m \neq 4)$), expression (36) obtains a simpler form:

$$D\{\hat{\beta}\} = \frac{\left(\frac{3}{2}P_{\mathrm{jn}} + P_{\mathrm{an}}\right)}{A^2K(N-1)^2\Psi^2\cos^2\beta}, \tag{37}$$

where $P_{\mathrm{jn}} = A^2\omega^2\sigma_\tau^2$ is average jitter noise power, $P_{\mathrm{an}} = 2\sigma\eta^2$ is average additive noise power.

When $m = 4$ and integral number of signal periods P expression (36) may be rewritten as follows:

$$D\{\hat{\beta}\} = \frac{1}{4P(N-1)^2\Psi^2\cos^2\beta}\left(\frac{P_{\mathrm{an}}}{A^2} + \frac{P_{\mathrm{jn}}}{A^2}\right.$$

$$\left.\times\left(\frac{3}{2} + \frac{(-1)^{P+n+1}}{8}\cos\big(4\varphi-2(N-1)\Psi\sin\beta\big)\cos\big(2(N-1)\Psi\sin\beta\big)\right)\right). \tag{38}$$

For the two-dimensional case of row-column array in a similar way we may obtain expressions for expected values and dispersions of variables α and β:

$$E\{\hat{\beta}\} = \beta,$$

$$E\{\hat{\alpha}\} = \alpha. \tag{39}$$

For dispersions of estimates we obtain the following expressions:

$$D\{\hat{\alpha}\} = \frac{3}{2}\frac{\omega^2\sigma_\tau^2}{K(N-1)^2 M\Psi^2\cos^2\alpha} + \frac{2\sigma_\eta^2}{A^2K(N-1)^2 M\Psi^2\cos^2\alpha}$$

$$+\frac{1}{2}\frac{\omega^2\sigma_\tau^2}{K^2(N-1)^2 M^2\Psi^2\cos^2\alpha}\frac{\sin(2\omega TK)}{\sin(2\omega T)}\frac{\sin(2\Upsilon M\sin\beta)}{\sin(2\Upsilon\sin\beta)}$$

$$\times\cos\left[2\omega T(K-1)+4\varphi-2\Psi(N-1)\sin\alpha-2\Upsilon(M-1)\sin\beta\right]$$

$$\times\cos\left(2\Psi(N-1)\sin\alpha\right), \tag{40}$$

$$D\{\hat{\beta}\} = \frac{3}{2}\frac{\omega^2\sigma_\tau^2}{KN(M-1)^2\Upsilon^2\cos^2\beta} + \frac{2\sigma_\eta^2}{A^2KN(M-1)^2\Upsilon^2\cos^2\beta}$$

$$+\frac{1}{2}\frac{\omega^2\sigma_\tau^2}{K^2N^2(M-1)^2\Upsilon^2\cos^2\beta}\frac{\sin(2\omega TK)}{\sin(2\omega T)}\frac{\sin(2\Psi N\sin\alpha)}{\sin(2\Psi\sin\alpha)}$$

$$\times\cos\left(2\Upsilon(M-1)\sin\beta\right)$$

$$\times\cos\left[2\omega T(K-1)+4\varphi-2\Psi(N-1)\sin\alpha-2\Upsilon(M-1)\sin\beta\right]. \tag{41}$$

Like in the previous case if the accumulation period contains an integral number of input signal's periods, i.e. when $fT = n/m$ is irreducible fraction (n and m are integers) and $K = Pm$ (P is integer, $(m > 2) \wedge (m \neq 4)$), expressions (40) and (41) obtain a simpler form:

$$D\{\hat{\alpha}\} = \frac{\frac{3}{2}P_{\text{jn}} + P_{\text{an}}}{A^2K(N-1)^2 M\Psi^2\cos^2\alpha},$$

$$D\{\hat{\beta}\} = \frac{\frac{3}{2}P_{\text{jn}} + P_{\text{an}}}{A^2KN(M-1)^2\Upsilon^2\cos^2\beta}. \tag{42}$$

SIMULATION RESULTS

To verify operation of precision estimates numerical simulation was carried out. In all experiments whose results are presented below the following conditions were maintained (except for the specially noted ones):

1) the amplitude of harmonic IF signal A equaled to 1000 quants of ADC;
2) the ration of IF to sampling frequency was 7/5;
3) the number of time samples K equaled to 5;
4) IF was 70 MHz;
5) mean-square deviation (MSD) for jitter σ_τ was expressed as a fraction of input signal's period;
6) MSD for additive noise σ_η was expressed in ADC quants;
7) distribution of additive noise and jitter samples was assumed to be normal;

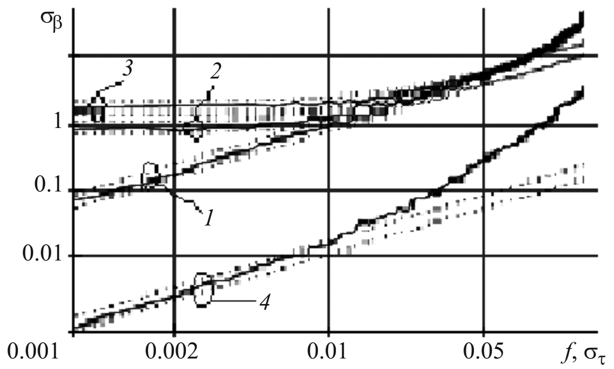

Fig. 3.

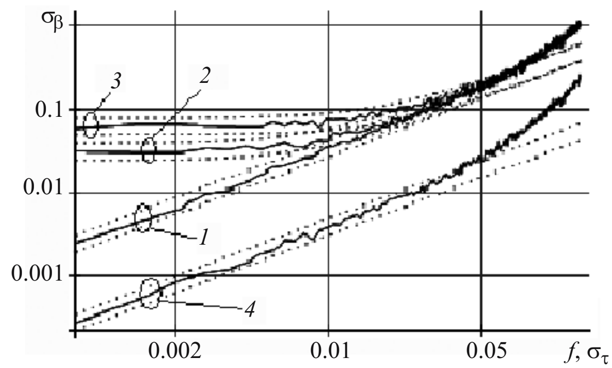

Fig. 4.

8) inter-element spacing along coordinate axis was assumed to equal half wavelength;

9) 100 realizations were used to obtain each point of the graph.

In Fig. 3 numerical experiments results for the 8-element linear DAR are presented. The dependence (37) of dispersion of signal's direction estimate on dispersion of jitter under constant dispersion of additive noise is illustrated. Direction to the signal source is $\beta = 20°$. The x axis specifies jitter MSD expressed as a fraction of IF period ($\sigma_\tau f$). The y axis denotes angle β estimate's MSD (in degrees). The logarithmic scale is used for both axes. The figure contains 4 numbered series of three curves are presented. Solid line denotes experimental MSD estimate obtained based on 100 realizations for each value of ($\sigma_\tau f$) (angle estimation was carried out using formula (12)). Dashed lines denote upper and lower boundaries of the trust interval. The trust interval is formed for the trust probability of 0.999 using χ^2 distribution. For 100 realizations a coefficient for the upper boundary of the trust interval amounts to 1.29. For the lower boundary the coefficient's value is 0.808 [7]. Series *1*, *2*, and *3* correspond to σ_η = 0, 50, 100, respectively. Series *4* corresponds to $\sigma_\eta = 0$ for linear DAR of 512 elements.

As follows from Fig. 3, for the mentioned experiment conditions for series *1–3* and ($\sigma_\eta f$) ≤ 0.05 MSD of angle β estimate accurate within the trust interval coincides with MSD calculated using formula (37). However when the number of elements in the antenna array exceeds 512 the value of ($\sigma_\eta f$), for which the experimental MSD lies within the trust interval decreases to ~0.01.

In Fig. 4 numerical experiments results for the 8×8-element row-column DAR are depicted. Dependence (42) is illustrated. Direction to the signal source is $\alpha = 20°$, $\beta = 20°$. The x axis specifies jitter MSD expressed as a fraction of IF period ($\sigma_\tau f$). The y axis denotes angle β estimate's MSD (in degrees). Since in the case of square array expressions for dispersion are identical accurate to the denotation, results for angle α are not presented. Both axes have the logarithmic scale. The figure contains 4 numbered series of three curves are presented. Solid line denotes experimental MSD estimate obtained based on 100 realizations for each value of ($\sigma_\tau f$) (angle estimation was carried out using formula (31)). Dashed lines denote upper and lower boundaries of the trust interval. The trust interval is formed for the trust probability of 0.999 using χ^2 distribution for 100 realizations. Series *1*, *2*, and *3* correspond to σ_η = 0, 50, 100, respectively. Series *4* corresponds to $\sigma_\eta = 0$ for the DAR with 32×32 elements.

As follows from Fig. 4, for series *1–3* and ($\sigma_\eta f$) ≤ 0.04 MSD of angle β estimate accurate within the trust interval coincides with MSD calculated using formula (37). When the number of elements in the antenna array exceeds 1024 (32×32), the value of ($\sigma_\eta f$), for which the experimental MSD lies within the trust interval decreases to ~0.03.

To compare the simulation results with real systems a 16-digit ADC AD9467 by Analog Devices with 250 MHz maximum sampling frequency and 900 MHz maximum analogue signal's bandwidth is considered. In this case the value of jitter MSD of 1 ps (the value achievable using inexpensive clock signal generators) with 900 MHz input harmonic signal's frequency corresponds to 9×10^{-4} of input signal's period, which lies within the applicability boundaries of the obtained expressions (37) and (42).

Infinitely increasing the value of amplitude A in expression (37) yields

$$\lim_{A \to +\infty} D\{\hat{\beta}\} = \frac{3}{2} \frac{\omega^2 \sigma_\tau^2}{K(N-1)^2 \Psi^2 \cos^2 \beta}. \tag{42}$$

Formula (42) serves as an analytical proof of results obtained using numerical simulations in work [2]:

1) direction-finding error caused by jitter may not be eliminated using increased energy;

2) the impact of jitter, like the impact of additive noise, may be decreased by increasing the length of signal's sample set and the number of elements in the antenna array.

Similar results may be obtained for the row-column array.

Expression (38) shows that if one uses a popular sampling method when the sampling frequency is an odd multiple of input signal frequency quarters, in the presence of ADC jitter dispersion for angle estimate depends on initial phase of the input signal. A similar conclusion may be obtained using expressions (40) and (41).

Due to the assumed infinitesimality of dispersion when obtaining results, the noise samples' distribution law is not specified.

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